CHAPTER 1

Truth Tables, Logic, and Proofs

1.1 STATEMENTS AND CONNECTIVES

In this section we develop truth tables and use them to begin the first step in logic. We will find as we continue in this chapter that truth tables are also a basic tool for other important concepts in discrete mathematics. Logic, developed by Aristotle (384–322 BCE), has been used throughout the centuries in the development of many areas of learning including theology, philosophy and mathematics. It is the foundation on which the whole structure of mathematics is built. Basically it is the science of reasoning, which allows us to determine which statements about mathematics are true and which are false based on a set of basic assumptions called axioms. Logic is also used in computer science to construct computer programs and to show that the programs do what they are designed to do. One of the primary goals of this book is to develop logic and show how to use it in computer science and to develop techniques for analyzing and proving theorems in mathematics.

A ***proposition*** is a statement or declarative sentence that may be assigned a true or false value. It must make sense to consider the statement being true or false. The true or false value assigned to a statement is called its ***truth value.***

Sentences that are not propositions include

Who are you?

(a question).

Read this chapter before the next class.

(a command or exclamation).

This sentence is not true.

(self contradictory).

We will use $p, q, r, \ldots$ to represent propositions. For example, p could represent the statement *It is going to rain tomorrow* and q could represent the statement *The square of an integer is positive.*

In English, sentences are combined using connectives and clauses to form more complex compound sentences. Common connectives are *and, or, not, if ... then,*

Case	p	q	r	p	$\vee$	$((\sim$	$q)$	$\wedge$	$r)$
1	T	T	T	T		F	T	$\boldsymbol{F}$	T
2	T	T	F	T		F	T	$\boldsymbol{F}$	F
3	T	F	T	T		T	F	$\boldsymbol{T}$	T
4	T	F	F	T		T	F	$\boldsymbol{F}$	F
5	F	T	T	F		F	T	$\boldsymbol{F}$	T
6	F	T	F	F		F	T	$\boldsymbol{F}$	F
7	F	F	T	F		T	F	$\boldsymbol{T}$	T
8	F	F	F	F		T	F	$\boldsymbol{F}$	F
				1		2	1	3	1

Finally the true values for $p \vee ((\sim q) \wedge r)$ are now placed under the $\vee$ symbol.

Case	p	q	r	p	$\vee$	$((\sim$	$q)$	$\wedge$	$r)$
1	T	T	T	T	$\boldsymbol{T}$	F	T	F	T
2	T	T	F	T	$\boldsymbol{T}$	F	T	F	F
3	T	F	T	T	$\boldsymbol{T}$	T	F	T	T
4	T	F	F	T	$\boldsymbol{T}$	T	F	F	F
5	F	T	T	F	$\boldsymbol{F}$	F	T	F	T
6	F	T	F	F	$\boldsymbol{F}$	F	T	F	F
7	F	F	T	F	$\boldsymbol{T}$	T	F	T	T
8	F	F	F	F	$\boldsymbol{F}$	T	F	F	F
				1	4	2	1	3	1

EXAMPLE 1.1

Let p represent the proposition *Fred likes football*, q represent the proposition *Fred likes golf*, and r represent the proposition *Fred likes tennis*. Translate the proposition *Fred likes football and it is not true that he likes golf or he likes tennis* to symbolic form and find the truth table.

First change the proposition to *Fred likes football and it is not true that Fred likes golf or Fred likes tennis. Fred likes golf or Fred likes tennis* is expressed symbolically as $q \vee r$. *It is not true that Fred likes golf or Fred likes tennis* is expressed symbolically as $\sim(q \vee r)$ because the negation applies to the entire "that" clause. Thus, the proposition is expressed as $p \wedge (\sim(q \vee r))$. Its truth table is

Case	p	q	r	p	$\wedge$	$(\sim$	$(q \vee r))$
1	T	T	T	T	$\boldsymbol{F}$	F	T
2	T	T	F	T	$\boldsymbol{F}$	F	T
3	T	F	T	T	$\boldsymbol{F}$	F	T
4	T	F	F	T	$\boldsymbol{T}$	T	F
5	F	T	T	F	$\boldsymbol{F}$	F	T
6	F	T	F	F	$\boldsymbol{F}$	F	T
7	F	F	T	F	$\boldsymbol{F}$	F	T
8	F	F	F	F	$\boldsymbol{F}$	T	F
				1	*	3	2

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EXERCISES

1. State which of the following are propositions. Give the truth value of the propositions.
   (a) What time is it?
   (b) The integer 1 is the smallest positive integer.
   (c) If $x = 3$, then $x^2 = 6$.
   (d) Look out for that car!
   (e) South Dakota is a southern state.
2. State which of the following are propositions. Give the truth value of the propositions.
   (a) All even numbers are divisible by 2.
   (b) Load the packages in the car.
   (c) This statement cannot possibly be true.
   (d) Jupiter is the closest planet to the sun.
   (e) Compact disks should never be stored in the microwave.
3. Let p, q, and r be the following statements:
   p : *Traveling to Mars is expensive.*
   q : *I will travel to Mars.*
   r : *I have money.*
   Express the following English sentences as symbolic expressions:
   (a) I have no money and I will not travel to Mars.
   (b) I have no money and traveling to Mars is expensive, or I would travel to Mars.
   (c) It is not true that I have money and will travel to Mars.
   (d) Traveling to Mars is not expensive and I will go to Mars, or traveling to Mars is expensive and I will not go to Mars.
4. Let p, q, and r be the following statements:
   p : *My computer is very fast.*
   q : *I will finish my project on time.*
   r : *I will pass the course.*
   Express the following English sentences as symbolic expressions:
   (a) My computer is not very fast or I would finish my project on time.
   (b) I will not finish my project on time and I will not pass the course.
   (c) It is not true that I will finish my project on time and pass the course.
   (d) My computer is very fast or I will not finish my project on time and pass the course.
5. Find the truth table for each sentence in Exercise 3.
6. Find the truth table for each sentence in Exercise 4.
7. Let p, q, and r be the following statements:
   p : *This game is very difficult.*
   q : *I play chess.*
   r : *It takes time to play chess.*
   Express the following symbolic expressions as English sentences:
   (a) $q \wedge r$

we form the proposition

$$(p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$$

This form of expression for a proposition is called ***disjunctive normal form***. The expressions $p \wedge q \wedge \sim r$, $p \wedge \sim q \wedge r$, and $\sim p \wedge q \wedge \sim r$ are called ***minterms***. More formally, we have the following definitions.

DEFINITION 1.5 If simple statements $p_1, p_2, p_3, \ldots, p_n$ are considered, then $x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_n$, where $x_i = p_i$ or $\sim p_i$, is called a ***minterm***. An expression which is the disjunction of minterms is in ***disjunctive normal form*** so that if $m_1, m_2, m_3, \ldots, m_n$ are minterms then $m_1 \vee m_2 \vee m_3 \vee \cdots \vee m_n$ is in disjunctive normal form.

Although every proposition can be expressed in disjunctive normal form, it is usually not the simplest form of the expression. In the next section, we will study Karnaugh maps which allow us to simplify expressions of propositions in disjunctive normal form.

In similar fashion, we note that $p \vee q \vee r$ is false only where p, q, and r are all false. In general, in the table

Case	p	q	r	
1	T	T	T	$\sim p \vee \sim q \vee \sim r$
2	T	T	F	$\sim p \vee \sim q \vee r$
3	T	F	T	$\sim p \vee q \vee \sim r$
4	T	F	F	$\sim p \vee q \vee r$
5	F	T	T	$p \vee \sim q \vee \sim r$
6	F	T	F	$p \vee \sim q \vee r$
7	F	F	T	$p \vee q \vee \sim r$
8	F	F	F	$p \vee q \vee r$

each expression is false on the line where it is located and true elsewhere. If we then want to find a statement having a given truth table, for each case where the truth table is false, we take the statement using that line on the table above corresponding to that case and connect these statements with $\wedge$. For example, to construct a proposition with the truth table

Case	p	q	r	
1	T	T	T	T
2	T	T	F	T
3	T	F	T	F
4	T	F	F	F
5	F	T	T	T
6	F	T	F	F
7	F	F	T	F
8	F	F	F	T