Exercises

1. (a) A G_δ *set* in a space X is a set A that equals a countable intersection of open sets of X. Show that in a first-countable T_1 space, every one-point set is a G_δ set.

(b) There is a familiar space in which every one-point set is a G_δ set, which nevertheless does not satisfy the first countability axiom. What is it?

The terminology here comes from the German. The "G" stands for "Gebiet," which means "open set," and the "δ" for "Durchschnitt," which means "intersection."

2. Show that if X has a countable basis $\{B_n\}$, then every basis $\mathcal{C}$ for X contains a countable basis for X. [*Hint:* For every pair of indices n, m for which it is possible, choose $C_{n,m} \in \mathcal{C}$ such that $B_n \subset C_{n,m} \subset B_m$.]

3. Let X have a countable basis; let A be an uncountable subset of X. Show that uncountably many points of A are limit points of A.

4. Show that every compact metrizable space X has a countable basis. [*Hint:* Let $\mathcal{A}_n$ be a finite covering of X by $1/n$-balls.]

5. (a) Show that every metrizable space with a countable dense subset has a countable basis.

(b) Show that every metrizable Lindelöf space has a countable basis.

6. Show that $\mathbb{R}_\ell$ and I_o^2 are not metrizable.

7. Which of our four countability axioms does S_Ω satisfy? What about $\bar{S}_\Omega$?

8. Which of our four countability axioms does $\mathbb{R}^\omega$ in the uniform topology satisfy?

9. Let A be a closed subspace of X. Show that if X is Lindelöf, then A is Lindelöf. Show by example that if X has a countable dense subset, A need not have a countable dense subset.

10. Show that if X is a countable product of spaces having countable dense subsets, then X has a countable dense subset.

11. Let $f : X \to Y$ be continuous. Show that if X is Lindelöf, or if X has a countable dense subset, then $f(X)$ satisfies the same condition.

12. Let $f : X \to Y$ be a continuous open map. Show that if X satisfies the first or the second countability axiom, then $f(X)$ satisfies the same axiom.

13. Show that if X has a countable dense subset, every collection of disjoint open sets in X is countable.

14. Show that if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

15. Give $\mathbb{R}^I$ the uniform metric, where $I = [0, 1]$. Let $\mathcal{C}(I, \mathbb{R})$ be the subspace consisting of continuous functions. Show that $\mathcal{C}(I, \mathbb{R})$ has a countable dense subset, and therefore a countable basis. [*Hint:* Consider those continuous functions whose graphs consist of finitely many line segments with rational end points.]

16. (a) Show that the product space $\mathbb{R}^I$, where $I = [0, 1]$, has a countable dense subset.

(b) Show that if J has cardinality greater than $\mathcal{P}(\mathbb{Z}_+)$, then the product space $\mathbb{R}^J$ does not have a countable dense subset. [*Hint:* If D is dense in $\mathbb{R}^J$, define $f : J \to \mathcal{P}(D)$ by the equation $f(\alpha) = D \cap \pi_\alpha^{-1}((a, b))$, where (a, b) is a fixed interval in $\mathbb{R}$.]

***17.** Give $\mathbb{R}^\omega$ the box topology. Let $\mathbb{Q}^\infty$ denote the subspace consisting of sequences of rationals that end in an infinite string of 0's. Which of our four countability axioms does this space satisfy?

***18.** Let G be a first-countable topological group. Show that if G has a countable dense subset, or is Lindelöf, then G has a countable basis. [*Hint:* Let $\{B_n\}$ be a countable basis at e. If D is a countable dense subset of G, show the sets dB_n, for $d \in D$, form a basis for G. If G is Lindelöf, choose for each n a countable set C_n such that the sets cB_n, for $c \in C_n$, cover G. Show that as n ranges over $\mathbb{Z}_+$, these sets form a basis for G.]

§31 The Separation Axioms

In this section, we introduce three separation axioms and explore some of their properties. One you have already seen—the Hausdorff axiom. The others are similar but stronger. As always when we introduce new concepts, we shall examine the relationship between these axioms and the concepts introduced earlier in the book.

Recall that a space X is said to be *Hausdorff* if for each pair x, y of distinct points of X, there exist disjoint open sets containing x and y, respectively.

Definition. Suppose that one-point sets are closed in X. Then X is said to be ***regular*** if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. The space X is said to be ***normal*** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

It is clear that a regular space is Hausdorff, and that a normal space is regular. (We need to include the condition that one-point sets be closed as part of the definition of regularity and normality in order for this to be the case. A two-point space in the indiscrete topology satisfies the other part of the definitions of regularity and normality, even though it is not Hausdorff.) For examples showing the regularity axiom stronger than the Hausdorff axiom, and normality stronger than regularity, see Examples 1 and 3.

These axioms are called separation axioms for the reason that they involve "separating" certain kinds of sets from one another by disjoint open sets. We have used the word "separation" before, of course, when we studied connected spaces. But in that case, we were trying to find disjoint open sets *whose union was the entire space*.

The present situation is quite different because the open sets need not satisfy this condition.

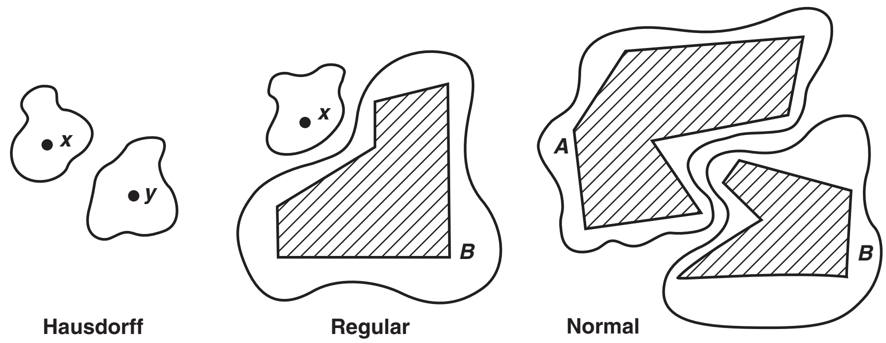

Figure 31.1

The three separation axioms are illustrated in Figure 31.1.

There are other ways to formulate the separation axioms. One formulation that is sometimes useful is given in the following lemma:

Lemma 31.1. *Let X be a topological space. Let one-point sets in X be closed.*

(a) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.

Proof. (a) Suppose that X is regular, and suppose that the point x and the neighborhood U of x are given. Let $B = X - U$; then B is a closed set. By hypothesis, there exist disjoint open sets V and W containing x and B, respectively. The set $\bar{V}$ is disjoint from B, since if $y \in B$, the set W is a neighborhood of y disjoint from V. Therefore, $\bar{V} \subset U$, as desired.

To prove the converse, suppose the point x and the closed set B not containing x are given. Let $U = X - B$. By hypothesis, there is a neighborhood V of x such that $\bar{V} \subset U$. The open sets V and $X - \bar{V}$ are disjoint open sets containing x and B, respectively. Thus X is regular.

(b) This proof uses exactly the same argument; one just replaces the point x by the set A throughout. ∎

Now we relate the separation axioms with the concepts previously introduced.

Theorem 31.2. *(a) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.*

(b) A subspace of a regular space is regular; a product of regular spaces is regular.

Proof. (a) This result was an exercise in §17. We provide a proof here. Let X be Hausdorff. Let x and y be two points of the subspace Y of X. If U and V are disjoint neighborhoods in X of x and y, respectively, then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y in Y.

Let $\{X_\alpha\}$ be a family of Hausdorff spaces. Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ be distinct points of the product space $\prod X_\alpha$. Because $\mathbf{x} \neq \mathbf{y}$, there is some index β such that $x_\beta \neq y_\beta$. Choose disjoint open sets U and V in X_β containing x_β and y_β, respectively. Then the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in $\prod X_\alpha$ containing $\mathbf{x}$ and $\mathbf{y}$, respectively.

(b) Let Y be a subspace of the regular space X. Then one-point sets are closed in Y. Let x be a point of Y and let B be a closed subset of Y disjoint from x. Now $\bar{B} \cap Y = B$, where $\bar{B}$ denotes the closure of B in X. Therefore, $x \notin \bar{B}$, so, using regularity of X, we can choose disjoint open sets U and V of X containing x and $\bar{B}$, respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y containing x and B, respectively.

Let $\{X_\alpha\}$ be a family of regular spaces; let $X = \prod X_\alpha$. By (a), X is Hausdorff, so that one-point sets are closed in X. We use the preceding lemma to prove regularity of X. Let $\mathbf{x} = (x_\alpha)$ be a point of X and let U be a neighborhood of $\mathbf{x}$ in X. Choose a basis element $\prod U_\alpha$ about $\mathbf{x}$ contained in U. Choose, for each α, a neighborhood V_α of x_α in X_α such that $\bar{V}_\alpha \subset U_\alpha$; if it happens that $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$. Then $V = \prod V_\alpha$ is a neighborhood of x in X. Since $\bar{V} = \prod \bar{V}_\alpha$ by Theorem 19.5, it follows at once that $\bar{V} \subset \prod U_\alpha \subset U$, so that X is regular. ∎

There is no analogous theorem for normal spaces, as we shall see shortly, in this section and the next.

EXAMPLE 1. *The space $\mathbb{R}_K$ is Hausdorff but not regular.* Recall that $\mathbb{R}_K$ denotes the reals in the topology having as basis all open intervals (a, b) and all sets of the form $(a, b) - K$, where $K = \{1/n \mid n \in \mathbb{Z}_+\}$. This space is Hausdorff, because any two distinct points have disjoint open intervals containing them.

But it is not regular. The set K is closed in $\mathbb{R}_K$, and it does not contain the point 0. Suppose that there exist disjoint open sets U and V containing 0 and K, respectively. Choose a basis element containing 0 and lying in U. It must be a basis element of the form $(a, b) - K$, since each basis element of the form (a, b) containing 0 intersects K. Choose n large enough that $1/n \in (a, b)$. Then choose a basis element about $1/n$ contained in V; it must be a basis element of the form (c, d). Finally, choose z so that $z < 1/n$ and $z > \max\{c, 1/(n+1)\}$. Then z belongs to both U and V, so they are not disjoint. See Figure 31.2.

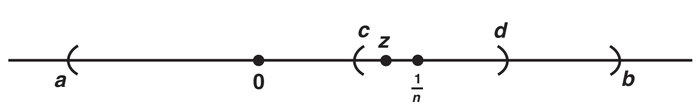

Figure 31.2